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Error estimates for nonlinear reaction-diffusion systems involving different diffusion length scales

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Abstract. We derive quantitative error estimates for coupled reaction-diffusion systems, whose coefficient functions are quasi-periodically oscillating modeling the microstructure of the underlying macroscopic domain. The coupling arises via nonlinear reaction terms and we allow for different diffusion length scales, i.e. whereas some species have characteristic diffusion length of order 1 other species may diffuse with the order of the characteristic microstructure-length scale. We consider an effective system, which is rigorously obtained via two-scale convergence, and we derive quantitative error estimates.

1. Introduction

Many mathematical models arising from biological, physical or engineering problems involve effects on microscopic scales, e.g. spatial inhomogeneities of the underlying material. In view of numerical simulations as well as more profound structural insight, we are interested in finding effective, or homogenized, models. From the analytical perspective, we ask for a rigorous justification of the effective model and, if available, error estimates describing the difference to the original macroscopic model.

We refer to the books [1, 2, 3, 4] for a general survey of homogenization theory. An important step in the theory of periodic homogenization was the introduction of two-scale convergence in [5, 6], which allows to rigorously treat systems involving different diffusion length scales, see e.g. [7, 8]. So far, the notion of two-scale convergence is a weak convergence. The periodic unfolding technique, introduced in [9], allows for a natural definition of strong two-scale convergence and, hence, the treatment of nonlinear problems, cf. [10, 11, 12, 13, 14, 15, 16]. Based on this strong notion of convergence, one can ask for quantitative error estimates, see e.g. [17, 18, 19, 20, 21], as well as for numerical simulations, see e.g. [22, 23, 24, 25, 26, 27]. For applications of periodic homogenization in physics and engineering, we refer to e.g. [28, 29, 30, 31] for systems of reaction-diffusion type in heterogeneous media as well as to e.g. [32, 33] for two-scale models on the evolution of damage.

The objective of this contribution are coupled reaction-diffusion systems of the following type

$$\begin{aligned} u_t^\varepsilon &= \operatorname{div}(\mathbb{D}_1(x, \frac{x}{\varepsilon}) \nabla u^\varepsilon) + F_1(x, \frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon) \\ v_t^\varepsilon &= \operatorname{div}(\varepsilon^2 \mathbb{D}_2(x, \frac{x}{\varepsilon}) \nabla v^\varepsilon) + F_2(x, \frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon) \end{aligned} \quad \text{in } \Omega \quad (1.1)$$

supplemented with homogeneous Neumann boundary conditions and initial conditions. Here, $(u^\varepsilon, v^\varepsilon) : [0, T] \times \Omega \rightarrow \mathbb{R}^{m_1+m_2}$ denote the concentrations of m_1 “classically” diffusing



species with characteristic diffusion length of order $O(1)$ and m_2 slowly diffusing species of order $O(\varepsilon)$. Moreover, $\mathbb{D}_i : \Omega \times \mathcal{Y} \rightarrow \mathbb{R}^{(m_i \times d) \times (m_i \times d)}$ denotes the diffusion coefficients and $F_i : \Omega \times \mathcal{Y} \times \mathbb{R}^{m_1+m_2} \rightarrow \mathbb{R}^{m_i}$ the nonlinear reaction terms and both, \mathbb{D}_i and F_i , are assumed to be periodic in $y = x/\varepsilon$ w.r.t. a prescribed microstructure.

It was shown in [34] that the solutions $(u^\varepsilon, v^\varepsilon)$ converge for $\varepsilon \rightarrow 0$ to a limit (u, V) that decomposes into a one-scale function $u(t, x)$ and a two-scale function $V(t, x, y)$, which solve the effective system

$$\begin{aligned} u_t &= \operatorname{div}(\mathbb{D}_{\text{eff}}(x) \nabla u) + \int_{\mathcal{Y}} F_1(x, y, u(x), V(x, y)) \, dy & \text{in } \Omega, \\ V_t &= \operatorname{div}_y(\mathbb{D}_2(x, y) \nabla_y V) + F_2(x, y, u, V) & \text{in } \Omega \times \mathcal{Y}. \end{aligned} \quad (1.2)$$

In order to install the limit passage (1.1) \rightarrow (1.2), we employ the technique of two-scale convergence via periodic unfolding, cf. (2.6). This involves the periodic unfolding operator $\mathcal{T}_\varepsilon : L^1(\Omega) \rightarrow L^1(\Omega \times \mathcal{Y})$, the folding operator $\mathcal{F}_\varepsilon : L^1(\Omega \times \mathcal{Y}) \rightarrow L^1(\Omega)$ and the gradient folding operators $\mathcal{G}_\varepsilon^0$ respective $\mathcal{G}_\varepsilon^1$, cf. Section 2.1. With this method, the strong two-scale convergence of the slowly diffusing species v^ε , i.e. $\max_{0 \leq t \leq T} \|\mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{L^2(\Omega \times \mathcal{Y})} \rightarrow 0$, was proved in [34], cf. Section 3.1, whereas the strong convergence $u^\varepsilon \rightarrow u$ follows immediately from the compact embedding $H^1(\Omega) \subset L^2(\Omega)$. This result was obtained under the assumption of L^∞ -regularity of the coefficients and global Lipschitz continuity of the reaction terms, cf. (3.6.A1)–(3.6.A4). One major analytical difficulty to overcome is the *periodicity defect* [17] or *\mathcal{T}_ε -property of recovered periodicity* [34], i.e.

$$\begin{aligned} \text{for all } u^\varepsilon \in H^1(\Omega) : \quad & \mathcal{T}_\varepsilon u^\varepsilon \in L^2(\Omega; H^1(\mathcal{Y})) \not\subset L^2(\Omega; H^1(\mathcal{Y})), \text{ but} \\ & \text{w-}\lim_{\varepsilon \rightarrow 0} \mathcal{T}_\varepsilon u^\varepsilon \in L^2(\Omega; H^1(\mathcal{Y})), \text{ if the limit exists.} \end{aligned} \quad (1.3.\text{PD})$$

The aim of this paper is to derive in Theorem 3.2 the error estimate

$$\max_{0 \leq t \leq T} \{ \|\mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{L^2(\Omega \times \mathcal{Y})} + \|u^\varepsilon(t) - u(t)\|_{L^2(\Omega)} \} \leq \varepsilon^{1/4} C. \quad (1.4)$$

In the interior of the domain Ω , the convergence rate in (1.4) can be improved to $\varepsilon^{1/2}$, see Theorem 3.3. We assume additional spatial regularity w.r.t. the macroscopic scale $x \in \Omega$ of the given data (3.6.A5), i.e. $\nabla_x \mathbb{D}_i, \nabla_x F_i \in L^\infty(\Omega \times \mathcal{Y})$, and the effective solution (u, V) (3.6.A6), i.e. $u \in H^2(\Omega)$, $V \in H^1(\Omega; H^1(\mathcal{Y}))$. We assume neither additional spatial regularity of the original solutions $(u^\varepsilon, v^\varepsilon)$ nor of the corrector functions.

In [20], a reaction-diffusion system predicting concrete corrosion is considered, but the system does not include slowly diffusing species v^ε . Nevertheless, for the classically diffusing species u^ε and its gradient ∇u^ε the convergence rate $\varepsilon^{1/2}$ and $\varepsilon^{1/4}$, respectively, is rigorously proved by the method of periodic unfolding. For systems involving slowly diffusing species v^ε , convergence rates of order $\varepsilon^{1/2}$ are derived in [24, 21] via the method of asymptotic expansion assuming continuous given data and limit solutions.

The distinctive feature of this contribution is the nonlinear coupling of the classically and slowly diffusing species combined with the periodic unfolding method, which allows to avoid any assumption of spatial continuity. Our proof to (1.4), in the first part, follows along the lines of [34] and we derive the Gronwall-type estimate

$$\frac{d}{dt} (\| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{L^2(\Omega \times \mathcal{Y})}^2 + \| u^\varepsilon - u \|_{L^2(\Omega)}^2) \leq C (\| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{L^2(\Omega \times \mathcal{Y})}^2 + \| u^\varepsilon - u \|_{L^2(\Omega)}^2) + \Delta^{v^\varepsilon} + \Delta^{u^\varepsilon}, \quad (1.5)$$

where $\| \cdot \| := \| \cdot \|_{L^2(\Omega \times \mathcal{Y})}$ and $\| \cdot \| := \| \cdot \|_{L^2(\Omega)}$ and $\Delta^{v^\varepsilon}, \Delta^{u^\varepsilon}$ comprise errors terms. In [34], it was shown that these errors vanish as $\varepsilon \rightarrow 0$. The novelty of this contribution, the second part of the proof, is the quantification of their convergence, namely $|\Delta^{u^\varepsilon} + \Delta^{v^\varepsilon}| \leq \varepsilon^{1/2} C$. In order to quantify those error terms, we have to find, in particular, error estimates for the folding and

unfolding operators, see the lemmas 3.5, 3.6, and 3.7 in Section 3.3, which heavily rely on the improved regularity w.r.t. $x \in \Omega$ and ideas from [17]. Moreover, we use a quantification result for the periodicity defect (1.3.PD) from [18], see Lemma 3.8.

The structure of the paper is as follows: In Section 2, we introduce basic notations, definitions, and results concerning periodic unfolding (Sec. 2.1) and two-scale convergence (Sec. 2.2). In Section 3, we consider the coupled systems (1.1)–(1.2) and derive the error estimate (1.4). Therefore, we list our assumptions and recall the existing convergence result (Sec. 3.1), state our Main Theorem (Thm. 3.2 & 3.3), explain the structure of its proof (Sec. 3.2), and we derive preparatory error estimates (Sec. 3.3). Finally, we give the proof of Theorem 3.2 (Sec. 3.4) and we discuss the obtained results (Sec. 3.5).

2. Two-scale convergence

Here, and throughout this paper, x denotes the macroscopic variable and the microscopic variable y captures periodic oscillations in x/ε . In order to describe the convergence from (1.1) to (1.2), we introduce the concept of two-scale convergence, which is designed for problems with underlying periodic microstructure. The definition of two-scale convergence (2.6), introduced in Section 2.2, is based on the periodic unfolding technique, described in Section 2.1, and with this it reduces to the notion of classical weak and strong convergence in the two-scale space $L^2(\Omega \times \mathcal{Y})$.

2.1. Periodic unfolding, folding, and gradient folding operators

Throughout this paper, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Following [9, 35, 13], $Y = [0, 1)^d$ denotes the *unit cell* so that \mathbb{R}^d is the disjoint union of translated cells $\lambda + Y$, where $\lambda \in \mathbb{Z}^d$. Identifying opposite faces of \bar{Y} gives the *periodicity cell* \mathcal{Y} , i.e. the torus

$$\mathcal{Y} := \mathbb{R}^d / \mathbb{Z}^d.$$

But, in notation, we will not distinguish between elements of the unit cell $y \in Y$ and the ones of the periodicity cell $y \in \mathcal{Y}$. Using the mappings $[\cdot]_Y : \mathbb{R}^d \rightarrow \mathbb{Z}^d$ and $\{\cdot\}_Y : \mathbb{R}^d \rightarrow Y$, we have the unique decomposition

$$\text{for all } x \in \mathbb{R}^d : x = [x]_Y + \{x\}_Y, \quad \text{where } [x]_Y \in \mathbb{Z}^d \text{ and } \{x\}_Y \in Y.$$

A function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ is called *Y-periodic*, if $f(x) = f(\{x\}_Y)$ for a.a. $x \in \mathbb{R}^d$. Then, we can identify every periodic function f with a function \tilde{f} on \mathcal{Y} . Introducing the small length scale parameter $\varepsilon > 0$, we define the sets

$$\Lambda_\varepsilon := \{\lambda \in \mathbb{Z}^d \mid \varepsilon(\lambda + Y) \subset \bar{\Omega}\} \quad \text{and} \quad \hat{\Omega}_\varepsilon := \text{int} \left(\bigcup_{\lambda \in \Lambda_\varepsilon} \varepsilon(\lambda + Y) \right).$$

With this definition of the subset $\hat{\Omega}_\varepsilon \subset \Omega$, we sort out microscopic cells $\varepsilon[x/\varepsilon]_Y + Y$ which overlap the boundary $\partial\Omega$. Moreover, we have $\text{vol}(\Omega \setminus \hat{\Omega}_\varepsilon) = O(\varepsilon)$ for those cells which are only partially contained in Ω . Based in these notations, the *periodic unfolding operator* $\mathcal{T}_\varepsilon : L^1(\Omega) \rightarrow L^1(\Omega \times \mathcal{Y})$ is defined via, cf. [9, 35],

$$(\mathcal{T}_\varepsilon u)(x, y) := \begin{cases} u \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{if } (x, y) \in \hat{\Omega}_\varepsilon \times \mathcal{Y}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Moreover, we have the crucial properties, cf. [35],

$$\begin{aligned} \text{product rule:} \quad & \mathcal{T}_\varepsilon(uv) = (\mathcal{T}_\varepsilon u)(\mathcal{T}_\varepsilon v) \quad \text{for all } u, v \in L^2(\Omega), \\ \text{unfolding criterion:} \quad & \int_\Omega F \, dx = \int_{\Omega \times \mathcal{Y}} \mathcal{T}_\varepsilon F \, dx \, dy + \omega_F(\varepsilon) \quad \text{for all } F \in L^1(\Omega), \end{aligned} \quad (2.2)$$

where $\omega_F(\varepsilon) = \int_{\Omega \setminus \widehat{\Omega}_\varepsilon} F \, dx$. It holds $\omega_F(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for all $F \in L^p(\Omega)$ with $p > 1$, due to $\text{vol}(\Omega \setminus \widehat{\Omega}_\varepsilon) \rightarrow 0$. The rate of $\omega(\varepsilon)$ depends on the norm of the function F .

For the reverse operation, we define the *folding operator* $\mathcal{F}_\varepsilon : L^1(\Omega \times \mathcal{Y}) \rightarrow L^1(\Omega)$ via

$$(\mathcal{F}_\varepsilon U)(x) := \int_{\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon Y} U(\xi, \left\{ \frac{x}{\varepsilon} \right\}_Y) \, d\xi \quad (2.3)$$

for all $x \in \widehat{\Omega}_\varepsilon$ and $(\mathcal{F}_\varepsilon U)(x) = 0$ otherwise.

Even for smooth functions $U : \Omega \times \mathcal{Y} \rightarrow \mathbb{R}$ the folded function $\mathcal{F}_\varepsilon U$ is only piecewise constant in x , hence $\nabla(\mathcal{F}_\varepsilon U)$ cannot be determined in the classical sense. Therefore, we define the so-called *gradient folding operator* $\mathcal{G}_\varepsilon^0$, respective $\mathcal{G}_\varepsilon^1$, which suitably regularizes the folded function $\mathcal{F}_\varepsilon U$. The definition of the above mentioned gradient folding operator is taken from [34, Def. 3.7], cf. also [16, Prop. 2.11], [10, Thm. 6.1], and [13, Prop. 2.10]. At first, we define the functions with zero average via

$$H_{\text{av}}^1(\mathcal{Y}) := \left\{ u \in H^1(\mathcal{Y}) \mid \int_{\mathcal{Y}} u(y) \, dy = 0 \right\}.$$

Definition 2.1 (Gradient folding). $\gamma = 0$: The gradient folding operator $\mathcal{G}_\varepsilon^0 : H^1(\Omega) \times L^2(\Omega; H_{\text{av}}^1(\mathcal{Y})) \rightarrow H^1(\Omega)$ maps a pair of functions $(u, U) \in H^1(\Omega) \times L^2(\Omega; H_{\text{av}}^1(\mathcal{Y}))$ to $u^\varepsilon := \mathcal{G}_\varepsilon^0(u, U)$, where $u^\varepsilon \in H^1(\Omega)$ is the unique weak solution of the elliptic problem

$$\int_{\Omega} (u^\varepsilon - u) \cdot \varphi + (\nabla u^\varepsilon - \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]) : \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1(\Omega). \quad (2.4)$$

$\gamma = 1$: The gradient folding operator $\mathcal{G}_\varepsilon^1 : L^2(\Omega; H^1(\mathcal{Y})) \rightarrow H^1(\Omega)$ maps a two-scale function $U \in L^2(\Omega; H^1(\mathcal{Y}))$ to $u^\varepsilon := \mathcal{G}_\varepsilon^1 U$, where $u^\varepsilon \in H^1(\Omega)$ is the unique weak solution of the elliptic problem

$$\int_{\Omega} (u^\varepsilon - \mathcal{F}_\varepsilon U) \cdot \varphi + (\varepsilon \nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)) : \varepsilon \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1(\Omega). \quad (2.5)$$

For $\varepsilon > 0$ fixed, the Lax-Milgram lemma yields the existence of a unique weak solution $u^\varepsilon \in H^1(\Omega)$ of (2.4)/(2.5), so that the gradient folding operators are indeed well-defined.

2.2. Weak and strong two-scale convergence

We are now in the position to give the definition of weak and strong two-scale convergence following again [9, 35, 13]. The notion of two-scale convergence was first introduced in [5] and coincides for bounded sequences with Definition (2.6a), here below. For a more detailed comparison of the different definitions see [13, Sec. 2.3].

For $(u^\varepsilon)_\varepsilon \subset L^2(\Omega)$, we say u^ε *weakly* (2.6a) respective *strongly* (2.6b) *two-scale converges* to U in $L^2(\Omega \times \mathcal{Y})$, if

$$u^\varepsilon \xrightarrow{2w} U \quad \text{in } L^2(\Omega \times \mathcal{Y}) \quad : \stackrel{\text{Def.}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u^\varepsilon \rightharpoonup U \quad \text{in } L^2(\Omega \times \mathcal{Y}), \quad (2.6a)$$

$$u^\varepsilon \xrightarrow{2s} U \quad \text{in } L^2(\Omega \times \mathcal{Y}) \quad : \stackrel{\text{Def.}}{\Longleftrightarrow} \quad \mathcal{T}_\varepsilon u^\varepsilon \rightarrow U \quad \text{in } L^2(\Omega \times \mathcal{Y}). \quad (2.6b)$$

The unfolding operator \mathcal{T}_ε is defined for the class of Lebesgue-integrable functions, where boundary values play no role, so that in particular $L^2(\Omega \times \mathcal{Y}) = L^2(\Omega \times Y)$. In view of the *periodicity defect* (1.3.PD), we carefully distinguish the spaces $H^1(Y)$ and $H^1(\mathcal{Y}) = H_{\text{per}}^1(Y)$, where the latter one is a closed subspace of $H^1(Y)$. For brevity, we set

$$\begin{aligned} X &= H^1(\Omega), \quad H = L^2(\Omega), \quad \mathbb{X} = L^2(\Omega; H^1(\mathcal{Y})), \\ \mathbb{X}_{\text{av}} &= L^2(\Omega; H_{\text{av}}^1(\mathcal{Y})), \quad \text{and} \quad \mathbb{H} = L^2(\Omega \times \mathcal{Y}). \end{aligned} \quad (2.7)$$

We have sequential compactness w.r.t. the weak two-scale convergence and it is shown in e.g. [5], [6, Prop. 1.14], [36, Thm. 5.2, Thm. 5.4], [7, Thm. 3.4] that bounded sequences of one-scale functions $(u^\varepsilon)_\varepsilon$ admit a weakly two-scale converging subsequence, i.e.

- (i) $\|u^\varepsilon\|_H \leq C \Rightarrow \exists U \in \mathbb{H} : u^{\varepsilon'} \xrightarrow{2w} U \text{ in } \mathbb{H},$
- (ii) $\|u^\varepsilon\|_H + \varepsilon \|\nabla u^\varepsilon\|_H \leq C \Rightarrow \exists U \in \mathbb{X} : u^{\varepsilon'} \xrightarrow{2w} U \text{ \& } \varepsilon' \nabla u^{\varepsilon'} \xrightarrow{2w} \nabla_y U \text{ in } \mathbb{H},$
- (ii) $\|u^\varepsilon\|_X \leq C \Rightarrow \exists (u, U) \in X \times \mathbb{X}_{av} : u^{\varepsilon'} \rightharpoonup u \text{ in } X \text{ and } \nabla u^{\varepsilon'} \xrightarrow{2w} \nabla u + \nabla_y U \text{ in } \mathbb{H}.$

Since (2.4) implies $\|\mathcal{G}_\varepsilon^1 U\|_H + \varepsilon \|\nabla(\mathcal{G}_\varepsilon^1 U)\|_H \leq C$, (ii) implies the existence of a weakly two-scale convergent subsequence. However, for given $U \in \mathbb{X}$ the gradient folding operator guarantees even *strong* two-scale convergence. So, $(\mathcal{G}_\varepsilon^1 U)_\varepsilon \subset X$ recovers any function $U \in \mathbb{X}$ via strong two-scale convergence and it is shown in [16, Prop. 2.11] that

$$\begin{aligned} \gamma = 0 : & \text{ for all } (u, U) \in X \times \mathbb{X}_{av} : \mathcal{G}_\varepsilon^0(u, U) \xrightarrow{2s} u \text{ \& } \nabla[\mathcal{G}_\varepsilon^0(u, U)] \xrightarrow{2s} \nabla u + \nabla_y U \text{ in } \mathbb{H}, \\ \gamma = 1 : & \text{ for all } U \in \mathbb{X} : \mathcal{G}_\varepsilon^1 U \xrightarrow{2s} U \text{ \& } \varepsilon \nabla[\mathcal{G}_\varepsilon^1 U] \xrightarrow{2s} \nabla_y U \text{ in } \mathbb{H}. \end{aligned}$$

Convenient commutation relations, such as $\mathcal{F}_\varepsilon(\nabla_y U) = \varepsilon \nabla(\mathcal{F}_\varepsilon U)$ or $\mathcal{G}_\varepsilon^1(\nabla_y U) = \varepsilon \nabla(\mathcal{G}_\varepsilon^1 U)$, cannot be expected, since $\mathcal{F}_\varepsilon U \notin X$ and $\nabla_y U \notin \mathbb{X}$. Instead, we have that the different folding operators are comparable in the sense that their difference vanishes, see [34, Prop. 3.9],

$$\begin{aligned} \gamma = 0 : & \text{ for all } (u, U) \in X \times \mathbb{X}_{av} : \\ & \|u - \mathcal{G}_\varepsilon^0(u, U)\|_H + \|\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla[\mathcal{G}_\varepsilon^0(u, U)]\|_H \rightarrow 0, \\ \gamma = 1 : & \text{ for all } U \in \mathbb{X} : \|\mathcal{F}_\varepsilon U - \mathcal{G}_\varepsilon^1 U\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1 U)\|_H \rightarrow 0. \end{aligned} \quad (2.8)$$

3. Error estimates for reaction-diffusion systems

We consider a system of two coupled reaction-diffusion systems, where the coupling arises via the nonlinear reaction term $(f_1^\varepsilon, f_2^\varepsilon)$, whereas the diffusion tensor has block structure

$$\begin{pmatrix} u_t^\varepsilon \\ v_t^\varepsilon \end{pmatrix} = \begin{pmatrix} \operatorname{div}(\mathbb{D}_1^\varepsilon \nabla u^\varepsilon) \\ \operatorname{div}(\varepsilon^2 \mathbb{D}_2^\varepsilon \nabla v^\varepsilon) \end{pmatrix} + \begin{pmatrix} f_1^\varepsilon(u^\varepsilon, v^\varepsilon) \\ f_2^\varepsilon(u^\varepsilon, v^\varepsilon) \end{pmatrix} \quad \text{in } [0, T] \times \Omega. \quad (3.1.P_\varepsilon)$$

We supplement (3.1.P_ε) with homogenous Neumann boundary conditions on $\partial\Omega$ and prescribed initial values $u^\varepsilon(0) = u_0^\varepsilon$ respective $v^\varepsilon(0) = v_0^\varepsilon$. In [34] (see Theorem 3.1 below) it was proven that $(u^\varepsilon, v^\varepsilon)$ converges for $\varepsilon \rightarrow 0$ to a limit (u, V) that decomposes into a one-scale function $u(t, x)$ and a two-scale function $V(t, x, y)$ which solve the effective system

$$\begin{pmatrix} u_t \\ V_t \end{pmatrix} = \begin{pmatrix} \operatorname{div}(\mathbb{D}_{\text{eff}} \nabla u) \\ \operatorname{div}_y(\mathbb{D}_2 \nabla_y V) \end{pmatrix} + \begin{pmatrix} f_{\text{eff}}(u, V) \\ F_2(u, V) \end{pmatrix} \quad \text{in } [0, T] \times \Omega \times \mathcal{Y}. \quad (3.2.P_0)$$

Here, the effective diffusion tensor \mathbb{D}_{eff} and the effective u -reaction f_{eff} only depend on the macroscopic variable $x \in \Omega$, while the diffusion tensor \mathbb{D}_2 and the V -reaction F_2 depend on the two-scale variables $(x, y) \in \Omega \times \mathcal{Y}$, see (3.6.A1)–(3.6.A2) and (3.3)–(3.5), below. The function-to-function map $f_{\text{eff}} : \Omega \times \mathbb{R}^{m_1} \times L^2(\mathcal{Y}; \mathbb{R}^{m_2}) \rightarrow \mathbb{R}^{m_1}$ is defined as

$$f_{\text{eff}}(x, u, Z) := \int_{\mathcal{Y}} F_1(x, y, u, Z(y)) \, dy. \quad (3.3)$$

The effective diffusion tensor $\mathbb{D}_{\text{eff}} : \Omega \rightarrow \mathbb{R}^{(m_1 \times d) \times (m_1 \times d)}$ is given componentwise via the classical homogenization formula, see e.g. [1, 6, 37],

$$\mathbb{D}_{\text{eff}}(x)_{ijkl} := \int_{\mathcal{Y}} \mathbb{D}_1(x, y)_{ijkl} + \sum_{r=1}^d \mathbb{D}_1(x, y)_{ijk r} \cdot \partial_{y_r} z(y)_{kl} \, dy, \quad (3.4)$$

for $i, k = 1, \dots, m_1$, $j, l = 1, \dots, d$, where the so-called correctors $z_{ij} \in H_{\text{av}}^1(\mathcal{Y})$ solve the local problem in the weak sense:

$$\operatorname{div}_y \left(\mathbb{D}_1(x, y)_{ijkl} + \sum_{r=1}^d \mathbb{D}_1(x, y)_{ijk r} \cdot \partial_{y_r} z(y)_{kl} \right) = 0 \quad \text{in } \mathcal{Y} \text{ for a.a. } x \in \Omega. \quad (3.5)$$

3.1. Assumptions and existing results

We recall the definition of the function spaces $(X, H, \mathbb{X}, \mathbb{X}_{\text{av}}, \mathbb{H})$ in (2.7) and we impose the following assumptions on the given data of (3.1.P $_{\varepsilon}$)–(3.2.P $_0$) for $i = 1, 2$:

The diffusion tensor

$$\begin{aligned} \mathbb{D}_i : \Omega \times \mathcal{Y} &\rightarrow \mathbb{R}^{(m_i \times d) \times (m_i \times d)} \text{ is uniformly bounded and elliptic, i.e.} \\ \exists \mu > 0 : \mathbb{D}_i(x, y) \xi : \xi &\geq \mu |\xi|^2 \text{ for all } \xi \in \mathbb{R}^{m_i \times d}, (x, y) \in \Omega \times \mathcal{Y}. \end{aligned} \quad (3.6.A1)$$

The reaction term

$$\begin{aligned} F_i : \Omega \times \mathcal{Y} \times \mathbb{R}^{m_1+m_2} &\rightarrow \mathbb{R}^{m_i} \text{ is uniformly bounded in } \Omega \times \mathcal{Y} \\ \text{as well as differentiable and globally Lipschitz continuous in } \mathbb{R}^{m_1+m_2}, \text{ i.e.} \\ \exists L > 0 : |F_i(x, y, A_1, B_1) - F_i(x, y, A_2, B_2)| &\leq L(|A_1 - A_2| + |B_1 - B_2|) \\ \text{for all } (A_i, B_i) \in \mathbb{R}^{m_1+m_2}, (x, y) \in \Omega \times \mathcal{Y}. \end{aligned} \quad (3.6.A2)$$

The initial values

$$\text{satisfy } u_0, \operatorname{div}(\mathbb{D}_{\text{eff}} \nabla u_0) \in H \text{ and } V_0, \operatorname{div}_y(\mathbb{D}_2 \nabla_y V_0) \in \mathbb{H}. \quad (3.6.A3)$$

The dependence on ε

$$\begin{aligned} \mathbb{D}_i^\varepsilon &:= \mathcal{F}_\varepsilon \mathbb{D}_i \text{ and } f_i^\varepsilon(\cdot, A, B) := \mathcal{F}_\varepsilon F_i(\cdot, \cdot, A, B) \text{ for all } (A, B) \in \mathbb{R}^{m_1+m_2}, \\ \exists c \geq 0 : \|u_0^\varepsilon\|_H + \|\operatorname{div}(\mathbb{D}_1^\varepsilon \nabla u_0^\varepsilon)\|_H + \|v_0^\varepsilon\|_H + \|\operatorname{div}(\varepsilon^2 \mathbb{D}_2^\varepsilon \nabla v_0^\varepsilon)\|_H &\leq c. \end{aligned} \quad (3.6.A4)$$

Spatial Lipschitz continuity of the given data

$$\begin{aligned} \text{For } (A, B) \in \mathbb{R}^{m_1+m_2} \text{ fixed, it holds } \nabla_x \mathbb{D}_i, \nabla_x F_i(A, B) &\in L^\infty(\Omega \times \mathcal{Y}) \\ \text{and we write } C_F &:= \sup_{(x, y) \in \Omega \times \mathcal{Y}} \{|F(x, y, A, B)| + |\nabla_x F(x, y, A, B)|\}. \end{aligned} \quad (3.6.A5)$$

Improved spatial regularity of the effective solutions

$$\forall t \in [0, T] : u(t) \in H^2(\Omega) \text{ and } V(t) \in H^1(\Omega; H^1(\mathcal{Y})), V_t(t) \in H^1(\Omega; L^2(\mathcal{Y})). \quad (3.6.A6)$$

Convergence rates for the initial values

$$\exists c \geq 0 : \|\mathcal{T}_\varepsilon v_0^\varepsilon - V_0\|_{\mathbb{H}} + \|u_0^\varepsilon - u_0\|_H \leq \varepsilon^{1/2} c. \quad (3.6.A7)$$

We obtain the two evolution triples $X \subset H \subset X^*$ and $\mathbb{X} \subset \mathbb{H} \subset \mathbb{X}^*$. The assumptions (3.6.A1)–(3.6.A4) guarantee the existence of unique weak solutions $(u^\varepsilon, v^\varepsilon)$ of (3.1.P $_{\varepsilon}$) and (u, V) of (3.2.P $_0$). Further, the differentiability of the reaction terms and the additional regularity of the initial values (3.6.A4) ensure improved time-regularity of the solutions and the following a priori bounds: there exists $C_b \geq 0$ independent of ε so that, cf. [34, Thm. 2.1 & Prop. 2.2],

$$\begin{aligned} \|u^\varepsilon\|_{C^1([0, T]; H)} + \|\nabla u^\varepsilon\|_{C([0, T]; H)} + \|v^\varepsilon\|_{C^1([0, T]; H)} + \varepsilon \|\nabla v^\varepsilon\|_{C([0, T]; H)} &\leq C_b, \\ \|u\|_{C^1([0, T]; H)} + \|\nabla u\|_{C([0, T]; H)} + \|V\|_{C^1([0, T]; \mathbb{H})} + \|\nabla_y V\|_{C([0, T]; \mathbb{H})} &\leq C_b. \end{aligned} \quad (3.7)$$

Moreover, we have the following convergence result.

Theorem 3.1 ([34, Thm. 5.1]). *Let the assumptions (3.6.A1)–(3.6.A4) as well as $u_0^\varepsilon \rightarrow u_0$ in H and $v_0^\varepsilon \xrightarrow{2s} V_0$ in \mathbb{H} be satisfied. The sequence of weak solutions $(u^\varepsilon, v^\varepsilon)$ of (3.1.P $_{\varepsilon}$) converges*

to the weak solution (u, V) of (3.2.P₀) in the following sense:

$$\max_{0 \leq t \leq T} \|\mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{\mathbb{H}} \rightarrow 0, \quad \varepsilon \nabla v^\varepsilon \xrightarrow{2s} \nabla_y V \text{ in } L^2(0, T; \mathbb{H}), \text{ and} \quad (3.8a)$$

$$v_t^\varepsilon \xrightarrow{2w} V_t \text{ in } L^2(0, T; \mathbb{H}), \text{ moreover } \forall t \in [0, T] : \varepsilon \nabla v^\varepsilon(t) \xrightarrow{2s} \nabla_y V(t) \text{ in } \mathbb{H};$$

$$u^\varepsilon \rightharpoonup u \text{ in } L^2(0, T; X) \text{ and } u_t^\varepsilon \rightharpoonup u_t \text{ in } H^1(0, T; X^*), \text{ moreover} \quad (3.8b)$$

$$\exists U \in L^2(0, T; \mathbb{X}_{av}) \text{ s.t. } \forall t \in [0, T] : \nabla u^\varepsilon(t) \xrightarrow{2w} \nabla u(t) + \nabla_y U(t) \text{ in } \mathbb{H}.$$

One may drop the additional assumptions $\operatorname{div}(\mathbb{D}_1^\varepsilon \nabla u_0^\varepsilon), \operatorname{div}(\varepsilon^2 \mathbb{D}_2^\varepsilon \nabla v_0^\varepsilon) \in H$ on the initial values, see [38]. Therein, it is shown that any solution with $u_0^\varepsilon, v_0^\varepsilon \in H$ can be approximated by a solution satisfying improved time-regularity as in (3.7).

3.2. Main Theorem and outline of the proof

Under the assumption of additional spatial regularity (3.6.A5)–(3.6.A7), we derive the following error estimates for the strong convergences in (3.8). We emphasize that we do not assume improved spatial regularity for the original macroscopic solutions $(u^\varepsilon, v^\varepsilon)$.

Theorem 3.2. *Let $(u^\varepsilon, v^\varepsilon)$ and (u, V) denote the solutions of (3.1.P_ε) and (3.2.P₀), respectively, and let the assumptions in (3.6) hold true. Then there exists a constant $C \geq 0$ independent of ε such that*

$$\max_{0 \leq t \leq T} \{\|\mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{\mathbb{H}} + \|u^\varepsilon(t) - u(t)\|_H\} \leq \varepsilon^{1/4} C, \quad (3.9a)$$

$$\|\mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon) - \nabla_y V\|_{L^2(0, T; \mathbb{H})} + \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}\|_{L^2(0, T; \mathbb{H})} \leq \varepsilon^{1/4} C. \quad (3.9b)$$

Moreover, we find the improved convergence rate in the interior of the domain Ω .

Theorem 3.3. *Let the assumptions (3.6) hold true. For all $\delta > 0$, let Ω_{int} denote an open subset of Ω with $\inf_{x \in \Omega_{\text{int}}} \operatorname{dist}(x, \partial\Omega) > \delta$. Then, there exists a constant $C_\delta \geq 0$ independent of ε such that for all $\varepsilon < \delta/(4\sqrt{d})$ it holds*

$$\begin{aligned} & \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{C([0, T]; L^2(\Omega_{\text{int}} \times \mathcal{Y}))} + \|\mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon) - \nabla_y V\|_{L^2(0, T; L^2(\Omega_{\text{int}} \times \mathcal{Y}))} \\ & + \|u^\varepsilon - u\|_{C([0, T]; L^2(\Omega_{\text{int}}))} + \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}\|_{L^2(0, T; L^2(\Omega_{\text{int}} \times \mathcal{Y}))} \leq \varepsilon^{1/2} C_\delta. \end{aligned}$$

Here, we focus on Theorem 3.2 and for the proof of Theorem 3.3, we refer to [38]. Therein, it is shown that away from the boundary $\partial\Omega$, the error $\sqrt{\varepsilon}$ of lower order does not need to be considered, cf. Lemma 3.4, and the periodicity defect error is of improved order ε using [17, Prop. 3.3 & Thm. 3.4].

Thanks to (3.6.A5), we can equally choose $\mathbb{D}^\varepsilon(x) = \mathbb{D}(x, x/\varepsilon)$ or $\mathbb{D}^\varepsilon = \mathcal{F}_\varepsilon \mathbb{D}$ in (3.6.A4) because we can identify $W^{1, \infty}(\Omega)$ with $C^{0, 1}(\Omega)$.

For $U \in L^2(0, T; \mathbb{X}_{av})$ in (3.8b) we have a.e. in $[0, T]$ the representation $U_i(x, y) = \sum_{j=1}^d \frac{\partial u_i}{\partial x_j}(x) z_{ij}(y)$, where the correctors $z_{ij} \in H_{av}^1(\mathcal{Y})$ solve the local problem (3.5). Since $u \in H^2(\Omega)$ by (3.6.A6), we obtain immediately $U \in H^1(\Omega; H_{av}^1(\mathcal{Y}))$ and in particular we do not assume any improved regularity for the correctors z_{ij} . Note, (3.9b) implies the strong two-scale convergence $\nabla u^\varepsilon \xrightarrow{2s} \nabla u + \nabla_y U$ in $L^2(0, T; \mathbb{H})$, which also holds in (3.8b) under the assumptions of Theorem 3.1.

Outline of the proof of Theorem 3.2: The essential idea is to derive the following Gronwall-type estimate

$$\frac{d}{dt} (\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2) \leq C \left(\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2 + \varepsilon^{1/2} \right). \quad (3.10)$$

Then, Gronwall's lemma yields for all $t \in [0, T]$

$$\|\mathcal{T}_\varepsilon v^\varepsilon(t) - V(t)\|_{\mathbb{H}}^2 + \|u^\varepsilon(t) - u(t)\|_H^2 \leq C \left(\|\mathcal{T}_\varepsilon v_0^\varepsilon - V_0\|_{\mathbb{H}}^2 + \|u_0^\varepsilon - u_0\|_H^2 + \varepsilon^{1/2} \right)$$

and using assumption (3.6.A7) on the initial values gives immediately (3.9a). We derive (3.10) in separate steps, namely

$$\frac{d}{dt} \|u^\varepsilon - u\|_H^2 \leq C \left(\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2 + \varepsilon^{1/2} \right) \text{ in Steps 1a-b, and} \quad (3.11)$$

$$\frac{d}{dt} \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 \leq C \left(\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2 + \varepsilon^{1/2} \right) \text{ in Steps 2a-b.} \quad (3.12)$$

- 1a. $\frac{d}{dt} \|u^\varepsilon - u\|_H^2$ -estimate: Following the argumentation in [34, Sect. 4.2/Proof of Thm. 4.1 (Step 2-5)], we derive the Gronwall-type estimate

$$\frac{d}{dt} \|u^\varepsilon - u\|_H^2 \leq C \left(\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2 + \Delta^{u^\varepsilon} \right), \quad (3.13)$$

where $\Delta^{u^\varepsilon} = \sum_{i=1}^5 |\Delta_i^{u^\varepsilon}|$ with $\Delta_1^{u^\varepsilon}$ (folding mismatch between \mathcal{F}_ε and $\mathcal{G}_\varepsilon^0$ resp. \mathcal{F}_ε and $\mathcal{G}_\varepsilon^1$)

$\Delta_2^{u^\varepsilon}$ (periodicity defect of \mathcal{T}_ε cf. (1.3.PD))

$\Delta_3^{u^\varepsilon}$ (approximation error $\mathbb{D}_1^\varepsilon \rightsquigarrow \mathbb{D}_{\text{eff}}$ resp. $\mathbb{D}_2^\varepsilon \rightsquigarrow \mathbb{D}_2$)

$\Delta_4^{u^\varepsilon}$ (approximation error $f_1^\varepsilon \rightsquigarrow f_{\text{eff}}$ resp. $f_2^\varepsilon \rightsquigarrow F_2$)

$\Delta_5^{u^\varepsilon}$ (unfolding error $\|V - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V\|_{\mathbb{H}}$ resp. $\|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}}$).

Above, $u \in H$ is canonically understood as two-scale function $u \in \mathbb{H}$. The last error term $\Delta_5^{u^\varepsilon}$ (resp. $\Delta_5^{v^\varepsilon}$) does not occur in [34], but is addressed as a one-liner here. Since $\frac{1}{2} \frac{d}{dt} \|u^\varepsilon - u\|_H^2 = \int_\Omega (u_t^\varepsilon - u_t) \cdot (u^\varepsilon - u) dx$, we ideally subtract the weak formulations of (3.1.P $_\varepsilon$)₁ and (3.2.P₀)₁ (resp. (3.1.P $_\varepsilon$)₂ and (3.2.P₀)₂), test with the difference $u^\varepsilon - u$ (resp. $\mathcal{T}_\varepsilon v^\varepsilon - V$) and we obtain (3.13). However, due to the two-scale structure of (3.2.P₀), analytical difficulties arise and we cannot proceed straight forward. We modify this basic idea as follows:

In the first step, we test (3.1.P $_\varepsilon$)₁ (resp. (3.1.P $_\varepsilon$)₂) with $u^\varepsilon - \mathcal{G}_\varepsilon^0(u, U)$ (resp. $v^\varepsilon - \mathcal{G}_\varepsilon^1 V$) and then, we *reformulate the ε -problem* into a two-scale problem using the unfolding operator \mathcal{T}_ε and the folding operators $\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon^0$ (resp. $\mathcal{G}_\varepsilon^1$). Due to regularity issues between \mathcal{F}_ε and $\mathcal{G}_\varepsilon^0$, cf. (2.8), we create the error term $\Delta_1^{u^\varepsilon}$ (resp. $\Delta_1^{v^\varepsilon}$).

In the second step, due to the periodicity defect (1.3.PD), we test (3.2.P₀)₁ (resp. (3.2.P₀)₂) only with (u, U) (resp. V). Afterwards, we *reformulate the limit problem* and insert the missing terms u^ε and $\mathcal{T}_\varepsilon(\nabla u^\varepsilon)$ (resp. $\mathcal{T}_\varepsilon v^\varepsilon$ and $\mathcal{T}_\varepsilon(\varepsilon \nabla v^\varepsilon)$) at the cost of creating the error term $\Delta_2^{u^\varepsilon}$ (resp. $\Delta_2^{v^\varepsilon}$).

Finally, in the third step, we add both reformulations and make further rearrangements in terms of the errors $\Delta_3^{u^\varepsilon} - \Delta_5^{u^\varepsilon}$ (resp. $\Delta_3^{v^\varepsilon} - \Delta_5^{v^\varepsilon}$) so that we end up with (3.13).

- 1b. *Estimation of Δ^{u^ε} and (3.11)*: We show $|\Delta^{u^\varepsilon}| \leq \varepsilon^{1/2} C$. In more detail, we apply Lemma 3.7 (with $\gamma = 0$) to $\Delta_1^{u^\varepsilon}$ and we use Lemma 3.8 (with $\gamma = 0$) for $\Delta_2^{u^\varepsilon}$. The remaining error terms $\Delta_3^{u^\varepsilon} - \Delta_5^{u^\varepsilon}$ resolve easily with Lemma 3.5 and (3.15).
- 2a. $\frac{d}{dt} \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2$ -estimate: In [34, Sect. 4.2/Proof of Thm. 4.1 (Step 2-5)], the following Gronwall-type estimate is proved

$$\frac{d}{dt} \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 \leq C \left(\|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2 + \|u^\varepsilon - u\|_H^2 + \Delta^{v^\varepsilon} \right), \quad (3.14)$$

where $\Delta^{v^\varepsilon} = \sum_{i=1}^5 |\Delta_i^{v^\varepsilon}|$.

- 2b. *Estimation of Δ^{v^ε} and (3.12):* We show $|\Delta^{v^\varepsilon}| \leq \varepsilon^{1/2}C$. As in Step 1b, we use Lemma 3.7 resp. Lemma 3.8 (with $\gamma = 1$) for $\Delta_1^{v^\varepsilon}$ resp. $\Delta_2^{v^\varepsilon}$ as well as Lemma 3.5 and (3.15) for $\Delta_3^{v^\varepsilon} - \Delta_5^{v^\varepsilon}$.
3. *Derivation of (3.9b):* We derive error estimates for the gradient terms by following the lines of [34, Proof of Thm. 4.1 (Step 7)].

3.3. Preparatory error estimates

We recall that Ω is a bounded domain with Lipschitz boundary such that we have in general $\widehat{\Omega}_\varepsilon \subsetneq \Omega$. With this, the treatment of cells $\varepsilon(\lambda_i + Y)$ intersecting the boundary $\partial\Omega$ is crucial. Therefore, we begin with a rather classical result for the error on $\Omega \setminus \Omega_\varrho$, where $\Omega_\varrho = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varrho\}$. The following lemma will be applied to the estimation of the boundary terms $\omega(\varepsilon)$ in (2.2) by choosing $\varrho = \varepsilon\sqrt{d}$.

Lemma 3.4 ([17, 18, 38]). *For $u \in X$ and $U \in H^1(\Omega; L^2(\mathcal{Y}))$, it holds for all $\varrho > 0$*

$$\|u\|_{L^2(\Omega \setminus \Omega_\varrho)} \leq (\varrho + \sqrt{\varrho})C\|u\|_X \quad \text{and} \quad \|U\|_{L^2(\Omega \setminus \Omega_\varrho \times \mathcal{Y})} \leq (\varrho + \sqrt{\varrho})C\|U\|_{H^1(\Omega; L^2(\mathcal{Y}))},$$

where the constant $C \geq 0$ only depends on the properties of the domain Ω .

The most important observation in deriving the error estimates (3.9a)–(3.9b) is the quantification of the well-known two-scale property, cf. [13, Prop. 2.4(e)], for every $U \in L^2(\Omega \times \mathcal{Y})$ exists a sequence $(u^\varepsilon)_\varepsilon \subset L^2(\Omega)$ such that $u^\varepsilon \xrightarrow{2s} U$ in $L^2(\Omega \times \mathcal{Y})$. For example, such a sequence is given by $u^\varepsilon = \mathcal{F}_\varepsilon U$. More precisely, based in the explicit definitions of \mathcal{T}_ε and \mathcal{F}_ε , it holds:

Lemma 3.5. *For all $U \in H^1(\Omega; L^2(\mathcal{Y}))$, there exists a constant $C \geq 0$, only depending on Ω and Y , such that*

$$\|U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U\|_{\mathbb{H}} \leq (\varepsilon + \varepsilon^{1/2})C\|U\|_{H^1(\Omega; L^2(\mathcal{Y}))}.$$

Proof. We use the unfolding criterion (2.2) and we apply the Poincaré-Wirtinger inequality on each cell $\text{int}(\varepsilon(\lambda_i + \varepsilon Y)) \subset \widehat{\Omega}_\varepsilon$ so that

$$\begin{aligned} \|U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U\|_{\mathbb{H}}^2 &= \sum_{\lambda_i \in \Lambda_\varepsilon} \int_{\varepsilon(\lambda_i + Y)} \int_{\mathcal{Y}} \left(U(x, y) - \int_{\mathcal{N}_\varepsilon(x) + \varepsilon Y} U(\xi, y) d\xi \right)^2 dx dy + \omega_U(\varepsilon) \\ &\leq \sum_{\lambda_i \in \Lambda_\varepsilon} C (\text{diam}(\varepsilon(\lambda_i + Y)))^2 \|\nabla_x U\|_{L^2(\lambda_i + \varepsilon Y)}^2 + \omega_U(\varepsilon) \leq \varepsilon^2 C \|U\|_{H^1(\Omega; L^2(\mathcal{Y}))}^2 + \omega_U(\varepsilon). \end{aligned}$$

Using Lemma 3.4 with $\varrho = \varepsilon\sqrt{d}$ gives

$$\omega_U(\varepsilon) = \int_{(\Omega \setminus \widehat{\Omega}_\varepsilon) \times \mathcal{Y}} (U - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U)^2 dx dy \leq ((\varepsilon + \sqrt{\varepsilon})C\|U\|_{H^1(\Omega; L^2(\mathcal{Y}))})^2.$$

Hence, we have the desired estimate. □

As a direct consequence of Lemma 3.5, we have, e.g. [17, Eq. (3.4)],

$$\text{for } u \in X : \quad \|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}} \leq (\varepsilon + \sqrt{\varepsilon})C\|u\|_X. \quad (3.15)$$

For possibly discontinuous functions $U \in H^1(\Omega; L^2(\mathcal{Y}))$, the “naive folding” $x \mapsto U(x, x/\varepsilon)$ is not well-defined. But, in the proof of Lemma 3.7 below, exactly such a “naive folding” is employed. Therefore, we need a suitable regularization U_ε of U so that $\vartheta_\varepsilon(x) = U_\varepsilon(x, x/\varepsilon)$ is well-defined and the difference $\|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H$ is of order $O(\varepsilon + \sqrt{\varepsilon})$. Therefore, we use in

addition to $\mathcal{G}_\varepsilon^0$ respective $\mathcal{G}_\varepsilon^1$ another regularization of the folding operator \mathcal{F}_ε , namely, the so-called *scale-splitting operator* \mathcal{Q}_ε , cf. [9, 35, 17].

For $u \in L^1(\Omega)$, the function $\mathcal{Q}_\varepsilon u$ is the \mathcal{Q}_1 -Lagrangian interpolant of the discrete function $\mathcal{F}_\varepsilon u$. Observe, $\mathcal{Q}_\varepsilon u \in W^{1,\infty}(\Omega)$ and $\mathcal{F}_\varepsilon u \in L^\infty(\Omega)$. (3.16)

Note, for general functions $u \in L^\infty(\Omega)$ and $z \in L^2(\mathcal{Y})$, the composition $x \mapsto u(x)z(x/\varepsilon)$ lies in $L^2(\Omega)$, see e.g. [37, Thm.4].

Lemma 3.6. For $w \in X$ and $z \in L^2(\mathcal{Y})$, there exists a constant $C \geq 0$, only depending on Ω and Y , such that

$$\|(\mathcal{F}_\varepsilon w - \mathcal{Q}_\varepsilon w) z(\cdot/\varepsilon)\|_H \leq \varepsilon^{1/2} C \|w\|_X \|z\|_{L^2(\mathcal{Y})}.$$

Proof. Based on the identity

$$\|(\mathcal{F}_\varepsilon w - \mathcal{Q}_\varepsilon w) z(\cdot/\varepsilon)\|_H^2 = \sum_{\lambda_i \in \Lambda_\varepsilon} \int_{\lambda_i + \varepsilon Y} |(\mathcal{F}_\varepsilon w(x) - \mathcal{Q}_\varepsilon w(x)) z(x/\varepsilon)|^2 dx + \omega_w(\varepsilon), \quad (3.17)$$

we consider in the following only one microscopic cell $\lambda_i + \varepsilon Y$, whereby w.l.o.g. $\lambda_i = 0$. The term $\omega_w(\varepsilon)$ comprises the boundary cells and it is treated with Lemma 3.4. By definition, we have for $x \in \varepsilon Y$ and every $\kappa = (\kappa_1, \dots, \kappa_d) \in \{0, 1\}^d$:

$$(\mathcal{Q}_\varepsilon w)(x) := \sum_{\kappa \in \{0,1\}^d} (\mathcal{F}_\varepsilon w)(\varepsilon \kappa) \cdot \bar{x}_1^{(\kappa_1)} \dots \bar{x}_d^{(\kappa_d)}, \text{ where } \bar{x}_l^{(\kappa_l)} := \begin{cases} \frac{x_l - \mathcal{N}_\varepsilon(x)_l}{\varepsilon} & \text{if } \kappa_l = 1, \\ 1 - \frac{x_l - \mathcal{N}_\varepsilon(x)_l}{\varepsilon} & \text{if } \kappa_l = 0. \end{cases}$$

With $\bar{x}_l^{(\kappa_l)} \in [0, 1]$, we obtain

$$\begin{aligned} \int_{\varepsilon Y} |(\mathcal{F}_\varepsilon w(0) - \mathcal{Q}_\varepsilon w(x)) z(x/\varepsilon)|^2 dx &\leq 2^d \sum_{\kappa \in \{0,1\}^d} |(\mathcal{F}_\varepsilon w(0) - \mathcal{F}_\varepsilon w(\varepsilon \kappa))|^2 \int_{\varepsilon Y} |z(x/\varepsilon)|^2 dx \\ &\leq 2^d \sum_{\kappa \in \{0,1\}^d} \left| \int_{\varepsilon Y} w(\xi) - w(\xi + \varepsilon \kappa) d\xi \right|^2 \varepsilon^d \|z\|_{L^2(\mathcal{Y})}^2 \leq 2^{2d} \varepsilon^2 d \|\nabla w\|_{L^2(\varepsilon Y)}^2 \|z\|_{L^2(\mathcal{Y})}^2. \end{aligned} \quad (3.18)$$

For the last estimate in (3.18), we use the fundamental relation $w(\xi) - w(\xi + \varepsilon \kappa) = \varepsilon \int_0^1 \nabla w(\xi + \varepsilon \kappa t) \cdot \kappa dt$ with $|\kappa| \leq \sqrt{d}$ so that we obtain for $|ds/d\xi| = 1$

$$\left| \int_{\varepsilon Y} w(\xi) - w(\xi + \varepsilon \kappa) d\xi \right|^2 \leq \varepsilon^2 d \int_{\varepsilon Y} \int_0^1 |\nabla w(\xi + \varepsilon \kappa t)|^2 dt d\xi = \frac{\varepsilon^2 d}{\varepsilon^d} \int_{\varepsilon Y} |\nabla w(s)|^2 ds.$$

Inserting (3.18) into (3.17) and summing up over all $\lambda_i \in \Lambda_\varepsilon$ gives the desired result. \square

The next Lemma quantifies the convergence (2.8) and relies on Lemma 3.6. It is applied to the estimation of the *folding mismatch* $\Delta_1^{u^\varepsilon}$ respective $\Delta_1^{v^\varepsilon}$.

Lemma 3.7. For all $(u, U) \in H^1(\Omega) \times H^1(\Omega; H_{\text{av}}^1(\mathcal{Y}))$ respective $U \in H^1(\Omega; H^1(\mathcal{Y}))$, there exists a constant $C \geq 0$ such that

$$\gamma = 0 : \quad \|\mathcal{G}_\varepsilon^0(u, U) - u\|_H + \|\nabla[\mathcal{G}_\varepsilon^0(u, U)] - \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]\|_H \leq \varepsilon^{1/2} C, \quad (3.19a)$$

$$\gamma = 1 : \quad \|\mathcal{G}_\varepsilon^1 U - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon \nabla[\mathcal{G}_\varepsilon^1 U] - \mathcal{F}_\varepsilon[\nabla_y U]\|_H \leq \varepsilon^{1/2} C. \quad (3.19b)$$

Proof. The proof follows in principle [16, Prop. 2.1]. It is adjusted to the estimate (3.19b) and it utilizes the gradient folding operator $\mathcal{G}_\varepsilon^1$ in the case $\gamma = 1$. In the case $\gamma = 0$, i.e. (3.19a), we resort to $\mathcal{G}_\varepsilon^0$ and we only point out the differences afterwards.

The case $\gamma = 1$: By an orthogonality argument, cf. [38], we may assume that

$$U(x, y) = w(x)z(y) \quad \text{with} \quad w \in X \text{ and } z \in H^1(\mathcal{Y}).$$

Recalling $\mathcal{G}_\varepsilon^1$ in (2.5) and \mathcal{Q}_ε in (3.16), we decompose $u^\varepsilon := \mathcal{G}_\varepsilon^1 U \in X$ as follows

$$u^\varepsilon(x) = \vartheta_\varepsilon(x) + g_\varepsilon(x) \quad \text{with} \quad \vartheta_\varepsilon(x) = \mathcal{Q}_\varepsilon w(x)z\left(\frac{x}{\varepsilon}\right). \quad (3.20)$$

By construction, we have $\vartheta_\varepsilon \in X$ and the remainder $g_\varepsilon \in X$ is defined for each $\varepsilon > 0$ as the solution of the elliptic problem

$$\begin{aligned} \int_{\Omega} g_\varepsilon \cdot \varphi + \varepsilon \nabla g_\varepsilon : \varepsilon \nabla \varphi \, dx &= \ell_\varepsilon(\varphi) \quad \text{for all } \varphi \in X, \text{ where} \\ \ell_\varepsilon(\varphi) &= \int_{\Omega} (\mathcal{F}_\varepsilon U - \vartheta_\varepsilon) \cdot \varphi + (\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon) : \varepsilon \nabla \varphi \, dx. \end{aligned} \quad (3.21)$$

The function g_ε can be estimated as follows

$$\begin{aligned} \frac{1}{2} (\|g_\varepsilon\|_H + \|\varepsilon \nabla g_\varepsilon\|_H)^2 &\leq \|g_\varepsilon\|_H^2 + \|\varepsilon \nabla g_\varepsilon\|_H^2 = \ell_\varepsilon(g_\varepsilon) \\ &\leq (\|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon\|_H) (\|g_\varepsilon\|_H + \|\varepsilon \nabla g_\varepsilon\|_H), \end{aligned} \quad (3.22)$$

which yields $\|g_\varepsilon\|_H + \|\varepsilon \nabla g_\varepsilon\|_H \leq 2(\|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_H + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon\|_H)$. Now, we estimate the difference between u^ε and $\mathcal{F}_\varepsilon U$ by adding and subtracting ϑ_ε . Recalling $g_\varepsilon = u^\varepsilon - \vartheta_\varepsilon$ and computing $\varepsilon \nabla \vartheta_\varepsilon = \varepsilon \nabla_x \vartheta_\varepsilon + \nabla_y \vartheta_\varepsilon$, we arrive at

$$\begin{aligned} &\|u^\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon \nabla u^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H \\ &\leq (\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|g_\varepsilon\|_H + \|\varepsilon \nabla \vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \|\varepsilon \nabla g_\varepsilon\|_H) \\ &\leq 3(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\varepsilon \nabla \vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H) \\ &\leq 3(\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\nabla_y \vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \varepsilon \|\nabla_x \vartheta_\varepsilon\|_H). \end{aligned} \quad (3.23)$$

According to [35, Prop. 4.5] it holds $\|\mathcal{Q}_\varepsilon w\|_X \leq C\|w\|_X$ and hence $\|\nabla_x \vartheta_\varepsilon\|_H \leq C\|\nabla_x U\|_H$. We proceed by estimating the remaining terms in (3.23) with the help of Lemma 3.6

$$\begin{aligned} &\|\vartheta_\varepsilon - \mathcal{F}_\varepsilon U\|_H + \|\nabla_y \vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H \\ &= \|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)z(\cdot/\varepsilon)\|_H + \|(\mathcal{Q}_\varepsilon w - \mathcal{F}_\varepsilon w)\nabla_y z(\cdot/\varepsilon)\|_H \leq \varepsilon^{1/2} C\|w\|_X \|z\|_{H^1(\mathcal{Y})} \end{aligned}$$

and thus (3.19b) is proved.

The case $\gamma = 0$: In (3.20), we set $u^\varepsilon := \mathcal{G}_\varepsilon^0(u, U)$ and decompose $u^\varepsilon = \eta_\varepsilon + g_\varepsilon$, where $\eta_\varepsilon = u + \varepsilon \vartheta_\varepsilon$ and $\vartheta_\varepsilon(x) = (\mathcal{Q}_\varepsilon w)(x)z(x/\varepsilon)$ for $U(x, y) = w(x)z(y)$.

In (3.21), we use $(g_\varepsilon, \varphi)_X = \ell_\varepsilon(\varphi)$ for all $\varphi \in X$ with $\ell_\varepsilon(\varphi) = \int_{\Omega} (u - \eta_\varepsilon) \cdot \varphi + (\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla \eta_\varepsilon) : \varepsilon \nabla \varphi \, dx$.

As in (3.22), we have $\|g_\varepsilon\|_H + \|\varepsilon \nabla g_\varepsilon\|_H \leq 2(\|u - \eta_\varepsilon\|_H + \|\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla \eta_\varepsilon\|_H)$.

In (3.23), we have $\nabla \eta_\varepsilon = \nabla u + \varepsilon \nabla_x \vartheta_\varepsilon + \nabla_y \vartheta_\varepsilon$ and hence $\|u^\varepsilon - u\|_H + \|\mathcal{F}_\varepsilon(\nabla u) - \nabla u\|_H + \|\nabla u^\varepsilon - \mathcal{F}_\varepsilon[\nabla u + \nabla_y U]\|_H \leq 3(\varepsilon \|\vartheta_\varepsilon\|_H + \|\nabla_y \vartheta_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)\|_H + \varepsilon \|\nabla_x \vartheta_\varepsilon\|_H)$. Again, the application of Lemma 3.5 & 3.6 and (3.15) as well as the improved regularity $(u, U) \in H^2(\Omega) \times H^1(\Omega; H_{\text{av}}^1(\mathcal{Y}))$ give (3.19a). \square

We use Lemma 3.8 below to estimate the *periodicity defect error* $\Delta_2^{u^\varepsilon}$ respective $\Delta_2^{v^\varepsilon}$.

Lemma 3.8 ([18, Thm.2.2 & 2.3]). *For every $u \in X$ with $\|u\|_X \leq c$ ($\gamma = 0$) and $\|u\|_H + \varepsilon\|\nabla u\|_H \leq c$ ($\gamma = 1$), there exists a function $\Psi_\varepsilon \in \mathbb{X}$ and $\Psi_\varepsilon \in \mathbb{X}_{\text{av}}$, respectively, and a constant $C \geq 0$, only depending on Ω and Y , such that*

$$\begin{aligned} \gamma = 0 : \quad & \|\Psi_\varepsilon\|_{\mathbb{X}} \leq C\|u\|_X \text{ and } \|\mathcal{T}_\varepsilon(\nabla u) - \{\nabla u + \nabla_y \Psi_\varepsilon\}\|_{L^2(Y; X^*)} \leq \varepsilon^{1/2} C\|u\|_X, \\ \gamma = 1 : \quad & \|\Psi_\varepsilon\|_{\mathbb{X}} \leq C(\|u\|_H + \varepsilon\|\nabla u\|_H) \text{ and} \\ & \|\mathcal{T}_\varepsilon u - \Psi_\varepsilon\|_{H^1(Y; X^*)} \leq \varepsilon^{1/2} C(\|u\|_H + \varepsilon\|\nabla u\|_H). \end{aligned}$$

3.4. Proof of Theorem 3.2

Proof of Theorem 3.2. By the uniform bounds (3.7), all functions are continuous in time and thus we can restore to work with estimates pointwise for all $t \in [0, T]$.

Step 1a: $\frac{d}{dt}\|u^\varepsilon - u\|_H^2$ -estimate. We refer to [38, Sect.2.1] for the complete proof of the Gronwall-type estimate (3.13). For simplicity in notation we suppress the index $i = 1$ so that the error terms $\Delta^{u^\varepsilon} := \sum_{i=1}^5 |\Delta_i^{u^\varepsilon}|$ take the precise formulations

$$\begin{aligned} \Delta_1^{u^\varepsilon} := & \int_{\Omega} (f^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon) \cdot (u - \mathcal{G}_\varepsilon^0(u, U)) \\ & - \mathbb{D}^\varepsilon \nabla u^\varepsilon : (\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla \mathcal{G}_\varepsilon^0(u, U)) \, dx, \end{aligned} \quad (3.24)$$

$$\Delta_2^{u^\varepsilon} := \int_{\Omega} (f_{\text{eff}}(u, V) - u_t) \cdot u^\varepsilon \, dx - \int_{\Omega \times \mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \, dx \, dy, \quad (3.25)$$

$$\Delta_3^{u^\varepsilon} := \int_{\Omega \times \mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon)[\nabla u + \nabla_y U] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u + \nabla_y U\}] \, dx \, dy, \quad (3.26)$$

$$\Delta_4^{u^\varepsilon} := \int_{\Omega} [f^\varepsilon(u, \mathcal{F}_\varepsilon V) - f_{\text{eff}}(u, V)] \cdot (u^\varepsilon - u) \, dx. \quad (3.27)$$

$$\Delta_5^{u^\varepsilon} = 2L\|V - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V\|_{\mathbb{H}}^2. \quad (3.28)$$

Step 1b: Estimation of Δ^{u^ε} and (3.11). We derive quantitative estimates of the errors $\Delta_1^{u^\varepsilon}, \dots, \Delta_5^{u^\varepsilon}$. We estimate the error $\Delta_1^{u^\varepsilon}$ in (3.24) with Lemma 3.7 and Lemma 3.5, viz.

$$\begin{aligned} |\Delta_1^{u^\varepsilon}| &= \left| \int_{\Omega} (f^\varepsilon(u^\varepsilon, v^\varepsilon) - u_t^\varepsilon) \cdot (u - \mathcal{G}_\varepsilon^0(u, U)) \, dx - \mathbb{D}^\varepsilon \nabla u^\varepsilon : [\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla \mathcal{G}_\varepsilon^0(u, U)] \, dx \right| \\ &\leq C(C_b) (\|u - \mathcal{G}_\varepsilon^0(u, U)\|_H + \|\mathcal{F}_\varepsilon[\nabla u + \nabla_y U] - \nabla \mathcal{G}_\varepsilon^0(u, U)\|_H) \\ &\leq \varepsilon^{1/2} C, \end{aligned} \quad (3.29)$$

where $C = C(C_b, \|U\|_{H^1(\Omega; H^1(\mathcal{Y}))}, \|u\|_{H^2(\Omega)})$ and we used (3.6.A2) and (3.7) to estimate the first integral.

We treat the second term $\Delta_2^{u^\varepsilon}$ in (3.25) with Lemma 3.8. There exists a two-scale function Ψ_ε so that $(u^\varepsilon, \Psi_\varepsilon) \in X \times \mathbb{X}_{\text{av}}$ is an admissible test function in the weak formulation of (3.2.P₀)₁ and hence

$$0 \equiv \int_{\Omega} (f_{\text{eff}}(u, V) - u_t) \cdot u^\varepsilon \, dx - \int_{\Omega \times \mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : [\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon] \, dx \, dy. \quad (3.30)$$

Subtracting (3.30) from (3.25) yields with Hölder's inequality and (3.6.A5)–(3.6.A6)

$$\begin{aligned} |\Delta_2^{u^\varepsilon}| &= \left| \int_{\Omega \times \mathcal{Y}} \mathbb{D}[\nabla u + \nabla_y U] : [\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon\}] \, dx \, dy \right| \\ &\leq \|\mathbb{D}[\nabla u + \nabla_y U]\|_{H^1(\Omega; L^2(\mathcal{Y}))} \|\mathcal{T}_\varepsilon(\nabla u^\varepsilon) - \{\nabla u^\varepsilon + \nabla_y \Psi_\varepsilon\}\|_{L^2(Y; X^*)} \\ &\leq \varepsilon^{1/2} C(C_b, \|\mathbb{D}\|_{W^{1,\infty}(\Omega; L^\infty(\mathcal{Y}))}, \|U\|_{H^1(\Omega; H^1(\mathcal{Y}))}). \end{aligned} \quad (3.31)$$

The third term $\Delta_3^{u^\varepsilon}$ in (3.26) is treated with Hölder's inequality and Lemma 3.5:

$$\begin{aligned} |\Delta_3^{u^\varepsilon}| &= \left| \int_{\Omega \times \mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon) [\nabla u + \nabla_y U] : [\nabla u + \nabla_y U - \mathcal{T}_\varepsilon (\nabla u^\varepsilon)] dx dy \right| \\ &\leq C(C_b) \|(\mathbb{D} - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon \mathbb{D}) [\nabla u + \nabla_y U]\|_{\mathbb{H}} \\ &\leq \varepsilon^{1/2} C(C_b, \|\mathbb{D}\|_{W^{1,\infty}(\Omega; L^\infty(\mathcal{Y}))}). \end{aligned} \quad (3.32)$$

The estimation of $\Delta_4^{u^\varepsilon}$ in (3.27) is a little more involved. Applying (2.2) only to the first term in (3.27) yields

$$\Delta_4^{u^\varepsilon} = \int_{\Omega \times \mathcal{Y}} \mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u) - F(u, V) \cdot (u^\varepsilon - u) dx dy.$$

Introducing the terms $\pm F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$ & $\pm F(u, V) \cdot \mathcal{T}_\varepsilon(u^\varepsilon - u)$, applying Hölder's inequality, and recalling the assumptions (3.7) & (3.6.A2) gives

$$\begin{aligned} |\Delta_4^{u^\varepsilon}| &\leq \|\mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}} \\ &\quad + \|F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(u, V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u)\|_{\mathbb{H}} \\ &\quad + \|F(u, V)\|_{\mathbb{H}} \|\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}} \\ &\leq C(L, C_F, C_b) (\|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V) - F(\mathcal{T}_\varepsilon u, \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V)\|_{\mathbb{H}} \\ &\quad + \|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V - V\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}}). \end{aligned} \quad (3.33)$$

$$+ \|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V - V\|_{\mathbb{H}} + \|\mathcal{T}_\varepsilon(u^\varepsilon - u) - (u^\varepsilon - u)\|_{\mathbb{H}}). \quad (3.34)$$

We exploit the Lipschitz continuity of F (3.6.A5) in (3.33) and we apply Lemma 3.5 resp. (3.15) in (3.34) so that we arrive at

$$|\Delta_4^{u^\varepsilon}| \leq \varepsilon^{1/2} C(L, C_b, C_F, \|V\|_{H^1(\Omega; L^2(\mathcal{Y}))}). \quad (3.35)$$

For the last error term we have immediately

$$|\Delta_5^{u^\varepsilon}| = 2L \|V - \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon V\|_{\mathbb{H}}^2 \leq \varepsilon C(L, \|V\|_{H^1(\Omega; L^2(\mathcal{Y}))}). \quad (3.36)$$

Recalling the Gronwall-type estimate (3.13), we combine the estimates (3.29), (3.31)–(3.32), (3.35)–(3.36) and hence we obtain the quantitative estimate (3.11).

Step 2a: $\frac{d}{dt} \|\mathcal{T}_\varepsilon v^\varepsilon - V\|_{\mathbb{H}}^2$ -estimate. For brevity we skip the index $i = 2$ in this step and the following. The proof of (3.14) can be found in [34, Proof of Thm. 4.1(Step2–5)], where the following error terms $\Delta^{v^\varepsilon} := \sum_{i=1}^5 |\Delta_i^{v^\varepsilon}|$ are derived

$$\Delta_1^{v^\varepsilon} := \int_{\Omega} (f^\varepsilon(u^\varepsilon, v^\varepsilon) - v_t^\varepsilon) \cdot (\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon^1 V) - \varepsilon \mathbb{D}^\varepsilon \nabla v^\varepsilon : [\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1 V)] dx, \quad (3.37)$$

$$\Delta_2^{v^\varepsilon} := \int_{\Omega \times \mathcal{Y}} [F(u, V) - V_t] \cdot \mathcal{T}_\varepsilon v^\varepsilon - \mathbb{D} \nabla_y V : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon) dx dy, \quad (3.38)$$

$$\Delta_3^{v^\varepsilon} := \int_{\Omega \times \mathcal{Y}} (\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon) \nabla_y V : \nabla_y(\mathcal{T}_\varepsilon v^\varepsilon - V) dx dy, \quad (3.39)$$

$$\Delta_4^{v^\varepsilon} := \int_{\Omega \times \mathcal{Y}} [\mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, V) - F(u, V)] \cdot (\mathcal{T}_\varepsilon v^\varepsilon - V) dx dy, \quad (3.40)$$

$$\Delta_5^{v^\varepsilon} := 2L \|\mathcal{T}_\varepsilon u - u\|_{\mathbb{H}}^2. \quad (3.41)$$

Step 2b: Estimation of Δ^{v^ε} and (3.12). Applying Lemma 3.7 to the first error term $\Delta_1^{v^\varepsilon}$ in (3.37) yields

$$|\Delta_1^{v^\varepsilon}| \leq C(C_b) (\|\mathcal{F}_\varepsilon V - \mathcal{G}_\varepsilon^1 V\|_H + \|\mathcal{F}_\varepsilon(\nabla_y V) - \varepsilon \nabla(\mathcal{G}_\varepsilon^1 V)\|_H) \leq \varepsilon^{1/2} C, \quad (3.42)$$

where $C = C(C_b, \|V\|_{H^1(\Omega; H^1(\mathcal{Y}))})$.

For the estimation of $\Delta_2^{v^\varepsilon}$ in (3.38), let $\Psi_\varepsilon \in \mathbb{X}$ be as in Lemma 3.8. Then, in particular, Ψ_ε is an admissible test function for $(3.2.P_0)_2$ and hence the application of Hölder's inequality and Lemma 3.8 gives

$$\begin{aligned} |\Delta_2^{v^\varepsilon}| &\leq \| |\mathbb{D} \nabla_y V| + |F(u, V)| + |V_t| \|_{H^1(\Omega; L^2(\mathcal{Y}))} \| \mathcal{T}_\varepsilon v^\varepsilon - \Psi_\varepsilon \|_{H^1(Y; X^*)} \\ &\leq \| |\mathbb{D} \nabla_y V| + |F(u, V)| + |V_t| \|_{H^1(\Omega; L^2(\mathcal{Y}))} \varepsilon C(\Omega) (\|v^\varepsilon\|_H + \varepsilon \|\nabla v^\varepsilon\|_H) \\ &\leq \varepsilon^{1/2} C, \end{aligned} \quad (3.43)$$

where $C = C(C_b, C_F, \|\mathbb{D}\|_{W^{1,\infty}(\Omega; L^\infty(\mathcal{Y}))}, \|V\|_{H^1(\Omega; L^2(\mathcal{Y}))}, \|V_t\|_{H^1(\Omega; L^2(\mathcal{Y}))})$.

Recalling $\mathbb{D}^\varepsilon = \mathcal{F}_\varepsilon \mathbb{D}$ and $f^\varepsilon = \mathcal{F}_\varepsilon F$, the error terms $\Delta_3^{v^\varepsilon} - \Delta_5^{v^\varepsilon}$ in (3.39)–(3.41) are estimated easily by using Lemma 3.5:

$$|\Delta_3^{v^\varepsilon}| \leq 2C_b \|(\mathbb{D} - \mathcal{T}_\varepsilon \mathbb{D}^\varepsilon) \nabla_y V\|_{\mathbb{H}} \leq \varepsilon^{1/2} C(C_b, \Omega, \|\mathbb{D}\|_{W^{1,\infty}(L^\infty(\mathcal{Y}))}), \quad (3.44)$$

$$|\Delta_4^{v^\varepsilon}| \leq 2C_b \| \mathcal{T}_\varepsilon f^\varepsilon(\mathcal{T}_\varepsilon u, V) - F(u, V) \|_{\mathbb{H}} \leq \varepsilon^{1/2} C(C_b, C_F), \quad (3.45)$$

$$|\Delta_5^{v^\varepsilon}| = 2L \| \mathcal{T}_\varepsilon u - u \|_{\mathbb{H}}^2 \leq \varepsilon C(L, \|u\|_X). \quad (3.46)$$

Overall the Gronwall-type estimate (3.14) and the quantitative estimates (3.42)–(3.46) give (3.12). Hence, we finish the proof of (3.9a) by applying Gronwall's lemma to (3.10).

Step 3: Derivation of (3.9b). According to [34, Proof of Thm. 4.1 (Step 7)], we have the estimate

$$\frac{1}{2} \frac{d}{dt} \| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{\mathbb{H}}^2 \leq -\mu \| \mathcal{T}_\varepsilon(\varepsilon \nabla u^\varepsilon) - \nabla_y V \|_{\mathbb{H}}^2 + 2L (\|u^\varepsilon - u\|_H^2 + \| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{\mathbb{H}}^2) + \Delta^{v^\varepsilon}.$$

Integrating over $[0, T]$ and exploiting (3.9a) as well as the estimations for Δ^{v^ε} in Step 2b yields

$$\begin{aligned} &\mu \| \mathcal{T}_\varepsilon(\varepsilon \nabla u^\varepsilon) - \nabla_y V \|_{L^2(0,T;\mathbb{H})}^2 \\ &\leq \int_0^T -\frac{1}{2} \frac{d}{dt} \| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{\mathbb{H}}^2 + 2L (\|u^\varepsilon - u\|_H^2 + \| \mathcal{T}_\varepsilon v^\varepsilon - V \|_{\mathbb{H}}^2) + |\Delta^{v^\varepsilon}| dt \leq T \varepsilon^{1/2} C. \end{aligned}$$

The estimation of the gradient follows analogously for the classically diffusing species u^ε . With this, we finish the proof of Theorem 3.2. \square

3.5. Discussion

We close the paper with a brief comparison of the obtained convergence rates. In [20], a nonlinearly coupled system of reaction-diffusion systems is considered on a cubical domain $\Omega \subset \mathbb{R}^3$ with exactly periodic, porous microstructure. The system does not include slowly diffusing species v^ε , but rather nonlinear boundary conditions at the surface of the pores. For the classically diffusing species u^ε the convergence rate $\varepsilon^{1/2}$ is rigorously proved by the method of periodic unfolding and results from [17, 18]. We emphasize that the gradient term is squared in [20, Thm 3.6], which means $\|\nabla u + \nabla_y U - \mathcal{T}_\varepsilon(\nabla u^\varepsilon)\|_{L^2(0,T;\mathbb{H})} \leq O(\varepsilon^{1/4})$. This error estimate is comparable with the one in Theorem 3.2. The focus of this text is the convergence of the slowly diffusing species v^ε which is strongly two-scale converging, as are $\varepsilon \nabla v^\varepsilon$ and ∇u^ε . In contrast, u^ε is strongly converging in $L^2(\Omega)$ and hence, the improved rate $\|u^\varepsilon - u\|_H \leq O(\varepsilon^{1/2})$ up to the boundary is not to expect for v^ε .

In [24, 21], nonlinearly coupled systems of reaction-diffusion equations involving diffusion length scales of order $O(1)$ and $O(\varepsilon)$ are considered in a heterogeneous setting. Whereas in [24] the coefficient functions are of the form $\mathbb{D}(x, x/\varepsilon)$, in [21], the heterogeneities in the domain $\Omega \subset \mathbb{R}^2$ are not arranged in a strictly periodic manner. In both cases, the approach of

formal asymptotic expansion is used and then, the convergence rate $O(\varepsilon^{1/2})$ is proved under the assumption of significantly more spatial regularity of the limit solution. In Theorem 3.3, our method reproduces the rate $O(\varepsilon^{1/2})$ as in [24, Thm. 4.5] and [21, Thm. 3.1] under significantly weaker assumptions on the given data and limit solutions.

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